PMM U.S.S.R.,Vol.46,pp.97—104
0021-8928/83/1 0097 $7.50/0

UDC 539.3

ON THE STATE OF STRESS OF AN ORTHOTROPIC ELASTIC PLANE REGION IN THE NEIGHBORHOOD OF A CORNER POINT*

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By using the solutions of certain special boundary value problems, the nature of the stress field features at a corner point is investigated. The asymptotic of the stress field is determined to constant accuracy in the neighborhood of the corner point, in particular, in the neighborhood of the terminus of a crack, a corner point of a re-entrant point type. As an example, there is considered the deformation of a half-plane with an oblique crack on the boundary. The features of stress fields at smooth points at which external loads have singularities, or the nature of the boundary conditions changes, are also determined.

The problem of corner points for an elliptic system of equations to which the system of statics equations of linear elasticity theory also belongs, was investigated in /1—3/, but re-entrant points were not considered. Results referring intrinsically to the system of elasticity theory equations for an isotropic body were published in /4/.

The results of this paper follow directly from the solutions of certain special problems /5/.

1. We write the governing relationships between the displacements u_j and the stresses p_{ij} as follows

$$\partial_j u_j = \sum_{k=1}^{2} a_{jjkk} p_{kk}, \quad \partial_1 u_2 + \partial_2 u_1 = 4a_{1212} p_{12}$$

where ∂_j is the operator of differentiation with respect to the coordinate $x_j, j = 1, 2$ (later we take $x_1 = x,\ x_2 = y$).

Here the homogeneity of the relationships means that the displacement field is defined just to the accuracy of an elementary three-dimensional field in the form of second-order polynomials, or the two-dimensional state of stress is generated by a two-dimensional displacement field.

The stresses, integral forces $P_k = \int p_k ds$ (ds is the differential of the arc contour of the domain D), and the displacements of the two-dimensional problem can be expressed in terms of two analytic functions $\Phi_n,\ n = 1, 2$ and their derivatives in the form /5/:

$$p_{jk} = 2\,\mathrm{Re} \sum_{n=1}^{2} (-\gamma_n)^{4-j-k} \Phi_n'(z_n), \quad P_k = -2\,\mathrm{Re} \sum_{n=1}^{2} (-\gamma_n)^{2-k} \Phi_n(z_n) \tag{1.1}$$

$$v_k = 2\,\mathrm{Re} \sum_{n=1}^{2} \gamma_n^{1-k} \varepsilon_{kn} \Phi_n(z_n), \quad \varepsilon_{kn} = a_{kk11}\gamma_n^2 + a_{kk22}$$

where $z_n = x + \gamma_n y = x_n + iy_n,\ \gamma_n = \alpha_n + i\beta_n\ (\beta_n > 0)$ are the roots of the characteristic equation $a_{1111}\gamma^4 + 2(a_{1122} + 2a_{1212})\gamma^2 + a_{2222} = 0$ where the case of equal roots is excluded.

The functions Φ_n are analytic in domains D_n that are images of the domain D in the mappings

$$z_n = A_n z, \quad A_n = \begin{Vmatrix} 1 & \alpha_n \\ 0 & \beta_n \end{Vmatrix}, \quad z = x + iy = \begin{Vmatrix} x \\ y \end{Vmatrix}$$

and the following representations are obtained /5/

$$\Phi_n = \frac{(-1)^{n-1}}{2t_n(\theta)} S_n v_n, \quad v_n = \frac{\varkappa_{3-n} P_\theta - u_{\theta^*}}{\varkappa_2 - \varkappa_1}, \quad n = 1, 2 \tag{1.2}$$

Here S_n is the Schwartz operator, recovering the function Φ_n by the boundary value of its real part $P_\theta = \int p_\theta ds$, p_θ and u_{θ^*} are projections of the forces and displacements in two mutually perpendicular directions determined by the slopes θ and θ^* to the x-axis, given as functions

*Prikl.Matem.Mekhan.,46,No.1,124-133,1982 of the point z of the boundary of the domain D so that

$$v_n(z)=v_n(A_n^{-1}z_n),\qquad t_n(\theta)=\gamma_n\cos\theta-\sin\theta,\quad \varkappa_n=(\varepsilon_{1n}\cos\theta^*+\gamma_n^{-1}\varepsilon_{2n}\sin\theta^*)t_n^{-1}(\theta)$$

The angles θ and θ^* are determined as a function of the kind of roots of the characteristic equation. If $\gamma_n=i\beta_n$, then $\theta=0,\ \theta^*=\pi/2$ or $\theta=\pi/2,\ \theta^*=0$, where both cases hold. If $\gamma_n=\pm\alpha+i\beta$ then

$$\operatorname{tg}\theta=\pm(a_{2222}/a_{1111})^{1/4},\qquad \operatorname{tg}\theta^*=\pm(a_{1111}/a_{2222})^{1/4}$$

For the purposes of the present paper it is sufficient to assume that the domain D is simply connected, and the unit circle or half-plane are mapped conformally in the domain D_n by using the function $f_n:\zeta_n\to z_n$. We then have formally

$$\Phi_n=\frac{(-1)^{n-1}}{2\pi i t_n(\theta)}\int\limits_L\frac{\psi_n(\tau_n)\,d\tau_n}{\tau_n-\zeta_n(z_n)}+c_n,\qquad \psi_n=v_n(A_n^{-1}f_n(\zeta_n))\tag{1.3}$$

where the constant c_n has an arbitrary imaginary part, and L is the unit circle or real interval $(-\infty,\infty)$.

Let us first prove the property of the mapping A_n needed later.

Lemma. Every angle in the mapping A_n less (greater) than π remains less (greater) than π.

Proof. Every line with the slope φ to the x-axis goes over in the mapping A_n into a line whose slope φ_n is defined by the formula

$$\operatorname{ctg}\varphi_n=(\operatorname{ctg}\varphi+\alpha_n)\beta_n^{-1},\quad \beta_n>0\tag{1.4}$$

Then every wedge with the angle $\nu\pi$ as the apex is mapped by using A_n into a wedge with angle $\nu_n\pi$, which is determined, as follows from (1.4), from the formula

$$\operatorname{ctg}(\varphi_n+\nu_n\pi)=(\operatorname{ctg}(\varphi+\nu\pi)+\alpha_n)\beta_n^{-1}\tag{1.5}$$

There remains just to note that both in the mapping $\varphi\to\varphi_n$ and in the mapping $\nu\to\nu_n$ defined, respectively, by (1.4) and (1.5) formulas, the points $0,\pi,\ 2\pi$ and 0,1,2 remain fixed. There are also other fixed points but the lemma is proved.

Let us consider the domain D with the angular point z_0 and let us make an assumption relative to the domain boundary that the function $z=z(s)$ (s is the arclength of the contour of the domain D) has a Hölder-continuous derivative from the left and right of z_0. Then as is known /6,7/, the function f that maps the unit circle conformally in a given domain, and its derivative have the form

$$z=f(\zeta)=z_0+(\zeta-\zeta_0)^\nu g(\zeta),\quad f'(\zeta)=(\zeta-\zeta_0)^{\nu-1}h(\zeta)\tag{1.6}$$

$\nu\pi$ is the angle between the tangents to the contour at the point $z_0\ (0<\nu\leqslant 2)$, the functions g and h are differentiable in a neighborhood of the point $\zeta_0=f^{-1}(z_0)$, and $g(\zeta_0)\neq 0,\ h(\zeta_0)=\nu g(\zeta_0)$ exist in continuity.

The linear mapping A_n does not alter the differential properties of the domain boundary to the left and right of the angular point, consequently it can be assumed that the representation

$$\begin{aligned}&z_n=f_n(\zeta_n)=(\zeta_n-\zeta_{n0})^{\nu_n}g_n(\zeta_n),\qquad f_n'(\zeta_n)=(\zeta_n-\zeta_{n0})^{\nu_n-1}h_n(\zeta_n)\\&\zeta_{n0}=f_n^{-1}(z_{n0}),\qquad h_n(\zeta_{n0})=\nu_n g_n(\zeta_{n0})\neq 0\end{aligned}\tag{1.7}$$

holds for the mapping of a unit circle into the domain D_n in the neighborhood of the corner point $z_{n0}=A_n z_0$, where ν_n is defined by (1.5), and by virtue of the lemma is included within the same limits as ν, i.e., $\nu_n\in(0,1)$ or $\nu_n\in(1,2)$, respectively.

With respect to p_θ and u_{θ^*}, considered here as functions of s, we assume that they are Hölder-continuous to the left and right of s_0, $z(s_0)=z_0$, where p_θ can have a finite discontinuity at s_0, a power-law singularity in s_0 is achieved for u'_{θ^*} (i.e., u'_{θ^*} belongs to the class H^* from /8/), and in the neighborhood of the point s_0 we give the representation

$$u_{\theta^*}=\begin{cases}u_0+(s-s_0)^\mu u_+(s), & s>s_0\\ u_0+(s_0-s)^\mu u_-(s), & s<s_0,\quad u_\pm(s_0)\neq 0\end{cases}\tag{1.8}$$

where $\mu>0$ and there exist finite unilateral derivatives $u_\pm'(s_0)$. For $\mu=1$ a finite discontinuity in s_0 is allowed for u'_{θ^*}.

Introducing the notation $w_n(s)=v_n(z(s))$, we have $\psi_n(\zeta_n(s))=w_n(s)$ according to (1.3) and

(1.2), where the function $\zeta_n(s)$ and its reciprocal are determined by the relationship $z_n(s) = f_n(\zeta_n)$ on the domain boundary. Differentiating this relationship, and taking account of (1.7), we obtain

$$\zeta_n'(s) = \frac{z_n'(s)}{h_n(\zeta_n)}(\zeta_n - \zeta_{n0})^{1-\nu_n}, \qquad \psi_n'(\zeta_n) = \frac{w_n(s)}{\zeta_n'(s)} \tag{1.9}$$

Let us note that $z_n'(s) = A_n z'(s) \neq 0$ since $|z'(s)| = 1$ and $\det A_n = \beta_n \neq 0$. Taking this remark into account, we determine the order of the singularity of the function ψ_n' as $\zeta_n \to \zeta_{n0}$.

We shall say that a certain quantity p has a singularity of order μ as $q \to q_0$ if for q sufficiently close to q_0 it is possible to determine $p = (q - q_0)^{\mu} a(q)$ and the function $a(q)$ is bounded in the neighborhood of q_0. For $\mu < 0$ we will call the singularity of zero order $-\mu$ as $q \to q_0$.

In the neighborhood of the point ζ_{n0} we introduce the local coordinate system $\zeta_n = \zeta_{n0} + \rho_n \exp(i\delta_n)$, where the angle δ_n is measured counter-clockwise from the tangent to the circle $|\zeta_n| = 1$ at the point ζ_{n0} so that $0 \leqslant \delta_n \leqslant \pi$. In the circle itself $\rho_n = 2\sin\delta_n$ and therefore, $d\zeta_n = 2\exp(i2\delta_n)d\delta_n$.

Furthermore, $z_n'(s) = z_n'(\delta_n)\, d\delta_n/ds$ and therefore we have $ds/d\delta_n = z_n'(\delta_n)/z_n'(s)$. According to (1.7), we compute

$$z_n'(\delta_n) = f_n'(\zeta_n)\, d\zeta_n/d\delta_n = 2^{\nu_n}(\sin\delta_n)^{\nu_n - 1}\exp(i(\nu_n - 1)\delta_n)\, h_n(\zeta_n)$$

and we then obtain $(d\delta_n/ds \geqslant 0)$:

$$\arg z_n'(s) = \arg z_n'(\delta_n) = (\nu_n + 1)\delta_n + \arg h_n(\zeta_n) \tag{1.10}$$

We determine the nature of the dependence of s on δ_n, and therefore, on ζ_n in the neighborhood of the point ζ_{n0} at both the left and right of this point, or equivalently, in the neighborhood $\delta_n = 0$ and $\delta_n = \pi$, respectively.

In the neighborhood of the point $\delta_n = 0$ we represent the difference $s - s_0$ $(s > s_0)$ in the form

$$s - s_0 = \int_0^{\delta_n} \left|\frac{z_n'(\delta_n)}{z_n'(s)}\right| d\delta_n = \int_0^{\delta_n} 2^{\nu_n}(\sin\delta_n)^{\nu_n - 1}\left|\frac{h_n(\zeta_n)}{z_n'(s)}\right| d\delta_n =$$

$$2^{\nu_n}\left|\frac{h_n(\zeta_{n0})}{z_n'(s_0 + 0)}\right| \int_0^{\delta_n} (\sin\delta_n)^{\nu_n - 1} d\sin\delta_n + o(\sin^{\nu_n}\delta_n), \qquad \delta_n \to 0$$

Summarizing, we will have as $s \to s_0 + 0$ $(\delta_n \to 0)$:

$$s - s_0 \approx \frac{1}{\nu_n}(2\sin\delta_n)^{\nu_n}\left|\frac{h_n(\zeta_{n0})}{z_n'(s_0 + 0)}\right| \tag{1.11}$$

Proceeding in an analogous manner, we obtain exactly the same expression for $s_0 - s$ $(s < s_0)$ in the neighborhood of the point $\delta_n = \pi$ as (1.11) with the replacement of $z_n'(s_0 + 0)$ by $z_n'(s_0 - 0)$ (we consider an analogous integral with the limits δ_n and π). Finally we conclude that $s - s_0$ and $s_0 - s$ are of zero order in ν_n as $\zeta_n \to \zeta_{n0}$.

Now we show that ψ_n' has a singularity of order $1 - \mu\nu_n$ as $\zeta_n \to \zeta_{n0}$. To do this we consider the following limit

$$\lim_{\zeta_n \to \zeta_{n0} \pm 0} (\zeta_n - \zeta_{n0})^{1-\mu\nu_n}\psi_n'(\zeta_n) = \lim (\zeta_n - \zeta_{n0})^{(1-\mu)\nu_n}\frac{w_n'(s)}{z_n'(s)}h_n(\zeta_n) = -\frac{h_n(\zeta_{n0})}{(\varkappa_2 - \varkappa_1)z_n'(s_0 \pm 0)}\lim(\zeta_n - \zeta_{n0})^{(1-\mu)\nu_n}u_{\theta*}'(s)$$

where relationships from (1.9) and (1.2) are used.

According to (1.8) and (1.11), we obtain

$$\lim_{\zeta_n \to \zeta_{n0} + 0} (\zeta_n - \zeta_{n0})^{(1-\mu)\nu_n}u_{\theta*}'(s) = \mu u_+(s_0)\lim(\zeta_n - \zeta_{n0})^{(1-\mu)\nu_n}(s - s_0)^{\mu - 1} =$$

$$\mu u_+(s_0)\lim_{\delta_n \to 0}(2\sin\delta_n\exp(i\delta_n))^{(1-\mu)\nu_n}\left(\frac{(2\sin\delta_n)^{\nu_n}}{\nu_n}\left|\frac{h_n(\zeta_{n0})}{z_n'(s_0 + 0)}\right|\right)^{\mu - 1}$$

and, finally taking into account that $\arg z_n'(s_0 + 0) = \arg h_n(\zeta_{n0})$ according to (1.10), we obtain

$$\lim_{\zeta_n \to \zeta_{n0} + 0}(\zeta_n - \zeta_{n0})^{1-\mu\nu_n}\psi_n'(\zeta_n) = -\frac{\mu u_+(s_0)}{\varkappa_2 - \varkappa_1}\nu_n^{1-\mu}\left|\frac{h_n(\zeta_{n0})}{z_n'(s_0 + 0)}\right|^{\mu} \tag{1.12}$$

Proceeding in an analogous manner as $\zeta_n \to \zeta_{n0} - 0$ and taking into account that $\arg z_n'(s_0 - 0) = (1 + \nu_n)\pi + \arg h_n(\zeta_{n0})$ according to (1.10), we obtain

$$\lim_{\zeta_n \to \zeta_{n0}-0} (\zeta_n - \zeta_{n0})^{1-\mu\nu_n} \psi_n'(\zeta_n) = -\exp(-i\mu\nu_n\pi) \frac{\mu u_-(s_0)}{\varkappa_2 - \varkappa_1} \nu_n^{1-\mu} \left| \frac{h_n(\zeta_{n0})}{z_n'(s_0-0)} \right|^{\mu} \tag{1.13}$$

Since $1 - \mu\nu_n < 1$, then the singularity for ψ_n' is integrable, and consequently, integration by parts is allowable for the evaluation of $\Phi_n'(z_n)$ in (1.3), resulting in the improper integral

$$\Phi_n' = \frac{(-1)^{n-1} I_n(\zeta_n)}{2\pi i t_n(\theta)} \frac{d\zeta_n}{dz_n}, \quad I_n(\zeta_n) = \int_L \frac{\psi_n'(\tau_n)\, d\tau_n}{\tau_n - \zeta_n}, \quad \zeta_n \in L \tag{1.14}$$

The results obtained below follow now from the properties and the asymptotic representation of a Cauchy type integral whose density has a weak singularity /8/.

Theorem 1. If $\mu < \min\{\nu_1^{-1}, \nu_2^{-1}, 1\}$ then the stress field has a singularity at the corner point.

The assertion of the theorem results from the asymptotic representation of the integral in (1.14) in the case $\mu < 1/\nu_n$. In this case, we have /8/ according to (1.12) and (1.13):

$$\Phi_n' = \frac{(-1)^{n-1} \mu \exp(-i\mu\varphi_{n0})}{i2(\varkappa_2 - \varkappa_1) t_n(\theta) \sin \mu\nu_n\pi (z_n - z_{n0})^{1-\mu}} \times \left(\frac{u_+(s_0) \exp(-i\mu\nu_n\pi)}{|z_n'(s_0+0)|^{\mu}} - \frac{u_-(s_0)}{|z_n'(s_0-0)|^{\mu}} \right) + \Psi_{n0}, \ \varphi_{n0} = \arg z_n'(s_0+0) \tag{1.5}$$

where $\Psi_{n0} \to 0$ as $z \to z_0$.

Since $1 - \mu > 0$, Φ_n' has a singularity, and then computing the stress by the first form from (1.1) for the principal term Φ_n' we see directly that all the stresses are not simultaneously zero, and therefore, the stress field has a singularity. The theorem is proved.

Let us consider two different kinds of corner points corresponding to the cases $\nu < 1$ and $\nu > 1$.

Theorem 2. If the internal angle at a point on the domain boundary is less than $\pi\ (\nu < 1)$, then the stresses at this point have a finite limit if and only if $\mu \geqslant 1$, i.e., when p_θ and $u_{\theta*}'$ are piecewise-Hölder-continuous.

Proof. If $\mu < 1$, then $\mu < \min\{\nu_1^{-1}, \nu_2^{-1}\}$ and by virtue of Theorem 1 the stress field has a singularity at the corner point. If $\mu \geqslant 1$, , then $1 - \mu\nu_n \leqslant 1 - \nu_n$ and ψ_n', and hence, the integral in (1.14) also has a singularity of order not greater than $1-\nu_n$. But since here $d\zeta_n/dz_n = 1/f_n'$ has a zero of order $1 - \nu_n$ according to (1.7), then consequently the Φ_n' are finite and the theorem is proved.

We present the asymptotic representations of the derivative for $\mu = 1$:

$$\Phi_n' = \frac{(-1)^{n-1} \exp(-i\varphi_{n0})}{2it_n(\theta) \sin \nu_n\pi} \left(\frac{w_n'(s_0-0)}{|z_n'(s_0-0)|} - \frac{w_n'(s_0+0)}{|z_n'(s_0+0)|} \right) + \Psi_{n0}$$

Substituting Φ_n' in (1.1), we obtain a representation for p_{jk}.

From the finiteness of the stress for $\nu < 1$ the finiteness of the force on the domain boundary follows by virtue of the Cauchy relationship

$$p_k = p_{k1}n_1 + p_{k2}n_2 \tag{1.16}$$

where (n_1, n_2) is the unit vector of the external normal to the boundary of the domain D.

Let us note that the converse is also true: the finiteness of the stress follows from the finiteness of the force on the boundary for $\nu < 1$. This results directly from the equilibrium conditions of specially isolated neighborhoods of the corner point. The result obtained agrees with the solution of the problem for an isotropic lens-shaped domain /9/.

For the case $\nu > 1$ the stress field can have a singularity at the corner point even for finite forces on the boundary. Let us examine this case.

2. If $\nu > 1$ and $\nu_n > 1$ according to the lemma, then from the condition $\mu < 1/\nu_n$ there results that $\mu < 1$ and then Theorem 1 is valid. If $\nu_n = 1/\mu$ for some n, then ψ_n' has no singularity. If here $\psi_n'(\zeta_{n0} - 0) = \psi_n'(\zeta_{n0} + 0)$ then Theorem 1 and the formula (1.15) are valid also in this case. If $\psi_n'(\zeta_{n0} - 0) \neq \psi_n'(\zeta_{n0} + 0)$ for $\nu_n = 1/\mu$, then the principal term in (1.15) acquires the factor $-\mu \ln(z_n - z_{n0})$, and Theorem 1 holds also in this case.

For $\mu > 1/\nu_n\ (n = 1, 2)$ the functions ψ_n' are Hölder-continuous, where $\psi_n'(\zeta_{n0} \pm 0) = 0$. By virtue of the Privalov–Plemelj theorem, the integral in (1.14) has the same continuity character. The singularity of Φ_n' is possible only because of the derivative $d\zeta_n/dz_n$, i.e., the order is not greater than $\nu_n - 1$ for $\zeta_n \to \zeta_{n0}$ or $1 - 1/\nu_n$ for $z_n \to z_{n0}$. Let the asymptotic equality hold here for $\zeta_n \to \zeta_{n0}$:

$$\frac{1}{\pi i}\int\limits_L \frac{\psi_n'(\tau_n)\,d\tau_n}{\tau_n-\zeta_n}\approx c_n(\zeta_n-\zeta_{n0})^{\mu_n},\quad \mu_n\geqslant 0,\quad \zeta_n\in L$$

Then there follows from (1.14) and (1.7)

$$\Phi_n'\approx\frac{(-1)^{n-1}c_n}{2t_n(\theta)\,\nu_n g_{n0}^{\lambda_n}(z_n-z_{n0})^{1-\lambda_n}},\quad \lambda_n=\frac{1+\mu_n}{\nu_n},\quad g_{n0}=g_n(\zeta_{n0}) \tag{2.1}$$

Substituting these derivatives ($n=1,2$) into (1.1), we obtain asymptotic representations for the stresses in the neighborhood of the corner point.

Let us examine the forces on the boundary. Being given the arbitrary direction at an angle ω to the x-axis, because of (1.16), (1.1), (1.14) and the Sokhotskii—Plemelj formula we obtain for the projection p_ω

$$p_\omega=(p_{11}y'(s)-p_{12}x'(s))\cos\omega+(p_{12}y'(s)-p_{22}x'(s))\sin\omega=2\operatorname{Re}\sum_{n=1}^{2}t_n(\omega)\Phi_n'(z_n)=$$
$$\operatorname{Re}\sum_{n=1}^{2}(-1)^{n-1}\sigma_n\left(\psi_n'(\zeta_n)+\frac{1}{i\pi}I_n(\zeta_n)\right)\frac{d\zeta_n}{ds},\quad \sigma_n=\frac{t_n(\omega)}{t_n(\theta)}$$

where, as mentioned in /5/, $\operatorname{Re}\sigma_1=\operatorname{Re}\sigma_2=\operatorname{Re}\sigma$. Since meanwhile, $\psi_n'(\zeta_n)\,\zeta_n'(s)=w_n'(s)$ are real, we then have

$$p_\omega=\operatorname{Re}\sigma p_\theta+\sum_{n=1}^{2}(-1)^{n-1}\operatorname{Re}\left(\frac{\sigma_n}{i\pi}I_n(\zeta_n)\frac{d\zeta_n}{ds}\right)$$

Setting $\omega=\theta$ here, we see that for $\sigma_n=1$ the sum is zero and hence

$$p_\omega=\operatorname{Re}\sigma p_\theta+\frac{1}{\pi}\sum_{n=1}^{2}(-1)^{n-1}\operatorname{Im}\sigma_n\operatorname{Re}\left(I_n(\zeta_n)\frac{d\zeta_n}{ds}\right) \tag{2.2}$$

The quantity p_ω is evidently finite if and only if the sum has a finite limit as $s\to s_0\pm 0$. In the case when both integrals in (2.2) differ from zero for $\zeta_n=\zeta_{n_0}$ the finiteness of the sum is possible only for identical orders of the singularities of the sum components. This latter is possible for suitable values of μ. For instance, if $\nu_1<\nu_2$ and $\mu\in(\nu_2^{-1},\nu_1^{-1})$, then from (1.15) and (2.1) it follows that μ should still satisfy the relationship $\mu=(1+\mu_2)/\nu_2$.

Of all the possible cases of the relationship for μ, let us just select the case when the corner point is a re-entrant point. In this case $\nu=2$ and $\nu_1=\nu_2=2$, where, because of Theorem 1, there remains to consider the case $\mu\geqslant {}^1/_2$. For $\mu={}^1/_2$ we assume that $\psi_n'(\zeta_{n0}-0)=\psi_n'(\zeta_{n0}+0)=a_n$ ($a_n=0$ for $\mu>{}^1/_2$).

According to (1.12) and (1.13), this is possible if $u_-(s_0)=u_+(s_0)$ and

$$a_n=-\frac{u_+(s_0)}{\varkappa_2-\varkappa_1}\left|\frac{g_n(\zeta_{n0})}{z_n'(s_0+0)}\right|^{1/2} \tag{2.3}$$

In the neighborhood of the corner point, we introduce a local coordinate system

$$z=z_0+r\exp(i\varphi),\quad \varphi_0\leqslant\varphi\leqslant\varphi_0+2\pi,\quad \varphi_0=\arg z'(s_0+0)=\arg g(\zeta_0)$$

We then have in the neighborhood of the point z_{n0}

$$z_n=A_n(z_0+r\exp(i\varphi))=z_{n0}+rk_n\exp(i\varphi_n),\qquad k_n=|\cos\varphi+\gamma_n\sin\varphi| \tag{2.4}$$
$$\varphi_{n0}\leqslant\varphi_n\leqslant\varphi_{n0}+2\pi,\quad \varphi_{n0}=\varphi_n(\varphi_0)=\arg z_n'(s_0+0)=\arg g_n(\zeta_{n0})$$

$\varphi_n=\varphi_n(\varphi)$ is determined from (1.4).

We obtain from the relationship $(\zeta_n-\zeta_{n0})^2g_n(\zeta_n)=rk_n\exp(i\varphi_n)$ which follows (1.7)

$$\frac{d\zeta_n}{dz_n}=\frac{1}{(\zeta_n-\zeta_{n0})h_n}\approx\frac{1}{2\sqrt{|g_{n0}|k_nr}}\exp\left(-i\frac{\varphi_n+\varphi_{n0}}{2}\right),\quad g_{n0}=g_n(\zeta_{n0}) \tag{2.5}$$

Substituting (2.5) into (1.14), and then in (1.1), we obtain the following asymptotic relationships for the stress tensor components in a polar coordinate system:

$$p_{lm}\approx\frac{1}{2\sqrt{r}}\sum_{n=1}^{2}\frac{(-1)^{n-1}}{\sqrt{|g_{n0}|k_n(\varphi)}}\times\operatorname{Re}\left(\frac{t_{nlm}(\varphi)}{t_n(\theta)}\left(a_n+\frac{b_n}{i\pi}\right)\exp\left(-i\frac{\varphi_n(\varphi)+\varphi_{n0}}{2}\right)\right) \tag{2.6}$$

$$lm=r\varphi,rr,\varphi\varphi;\quad t_{nr\varphi}=-(\gamma_n\cos\varphi-\sin\varphi)(\gamma_n\sin\varphi+\cos\varphi)$$
$$t_{nrr}=(\gamma_n\cos\varphi-\sin\varphi)^2,\quad t_{n\varphi\varphi}=(\gamma_n\sin\varphi+\cos\varphi)^2;\ b_n=I_n(\zeta_{n0})$$

It follows from (2.3) that a_n are real. The quantities b_n also turn out to be real.

If $L = (-\infty, \infty)$, then b_n are evidently real since in this case all the variables are real in $I_n(\zeta_{n0})$ under the integral. In the case when L is the unit circle, it follows from the geometric constructions that

$$(\tau_n - \zeta_{n0})^{-1} = \frac{1}{2}(\mathrm{ctg}\,\delta_n(s) - i)$$

$\delta_n(s)$ is the angle between the chord $\tau_n - \zeta_{n0}$ and the tangent at a point ζ_{n0} measured counter-clockwise from the tangent. And since $\psi_n'(\tau_n)\,d\tau_n = w_n'(s)ds$, we then obtain

$$b_n = \frac{1}{2}\int_0^l (\mathrm{ctg}\,\delta_n(s) - i)\,w_n'(s)\,ds = \frac{1}{2}\int_0^l \mathrm{ctg}\,\delta_n(s)\,w_n'(s)\,ds$$

because of the continuity of $w_n(s)$ and the periodicity $w_n(l) = w_n(0)$ (l is the length of the contour of the domain D). Therefore, b_n are proved to be real.

The integral representations (2.3) agree in the nature of their dependence on r with the representations for cracks in an unlimited body /10/, except here the intensity coefficients are determined in terms of a_n and b_n.

Let us consider the condition for finiteness of the force on the boundary for the stress field (2.6). Since $z'(s_0+0) = \exp(i\varphi_0)$, then $z_n'(s_0+0) = k_n(\varphi_0)\exp(i\varphi_{n0})$ and $z_n'(s_0-0) = k_n(\varphi_0 + \pi)\exp(i\varphi_{n0} + i\pi)$, $k_n(\varphi_0) = |\cos\varphi_0 + \gamma_n \sin\varphi_0|$. It then follows from (2.5) that

$$\frac{d\zeta_n}{ds} = \frac{d\zeta_n}{dz_n} z_n'(s) \approx \frac{1}{2}\sqrt{\frac{k_n(\varphi_0)}{|g_{n0}|r}}, \qquad r \to 0$$

Substituting into (2.2) and taking into account that the b_n are real, we obtain

$$p_\omega \approx \frac{1}{2\sqrt{r}}\sum_{n=1}^{2}(-1)^{n-1} b_n \,\mathrm{Im}\,\sigma_n \sqrt{\frac{k_n(\varphi_0)}{|g_{n0}|}} \tag{2.7}$$

It is hence seen that the sum should be zero for the finiteness of p_ω, whereupon we have (if $b_n \neq 0$):

$$b_n = \varepsilon_n b, \qquad \varepsilon_n = \mathrm{Im}\,\sigma_n \sqrt{k_n(\varphi_0)/|g_{n0}|}$$

The constant b is determined from the solution of the problem as a whole.

Example. We consider the deformation of a half-plane $(y > 0)$ with a crack on the boundary. Let $\lambda\pi$ be the slope of the crack to the half-plane boundary, measured counter-clockwise from the boundary, and l the crack length. Then the function mapping the half-plane $\eta = \mathrm{Im}\,\zeta > 0$ onto the half-plane with a crack has the form /11/:

$$z = c(\zeta + 1 - \lambda)^{1-\lambda}(\zeta - \lambda)^{\lambda}, \quad c = l(1-\lambda)^{-1+\lambda}\lambda^{-\lambda}, \quad 0 < \lambda < 1$$

The origin here $\zeta_0 = 0$ is mapped at the end of the crack, at the point $z_0 = l\exp(i\lambda\pi)$. The mapping A_n transfers the half-plane with the crack, inclined at an angle $\lambda\pi$ to the boundary, into a half-plane with a crack inclined at an angle $\lambda_n\pi$ to the boundary, determined by (1.4). Then the mapping of the half-plane in the new domain will be realized by the functions

$$z_n = c_n(\zeta_n + 1 - \lambda_n)^{1-\lambda_n}(\zeta_n - \lambda_n)^{\lambda_n}, \; c_n = l_n(1-\lambda_n)^{-1+\lambda_n}\lambda_n^{-\lambda_n}$$

where $l_n = k_n(\lambda\pi)$, k_n from (2.4), here the origin $\zeta_{n0} = 0$ is mapped into the end of the crack $z_{n0} = l_n \exp(i\lambda_n\pi)$.

In order to determine the asymptotic of the stress field in the neighborhood of the crack tip, we expand the function $z_n = f_n(\zeta_n)$ in the neighborhood of $\zeta_{n0} = 0$ in a power series in ζ_n:

$$z_n = l_n \exp(i\lambda_n\pi)\left(1 - \frac{\zeta_n^2}{2\lambda_n(1-\lambda_n)} + \ldots\right)$$

Comparing with (1.7), we obtain $g_{n0} = l_n \exp(-i(1-\lambda_n)\pi)(2\lambda_n(1-\lambda_n))^{-1}$. Then subsituting into (2.6), we obtain for instance

$$p_{\varphi\varphi} \approx \frac{1}{2\sqrt{r}}\sum_{n=1}^{2}(-1)^{n-1}\left(\frac{\lambda_n(1-\lambda_n)}{l_n k_n(\varphi)}\right)^{1/2} \times \mathrm{Re}\left(\frac{(\gamma_n \sin\varphi + \cos\varphi)^2}{t_n(\theta)}\left(ia_n + \frac{b_n}{\pi}\right)\exp\left(-i\,\frac{\varphi_n(\varphi) + \lambda_n\pi}{2}\right)\right)$$

and the condition for finiteness of the forces on the boundary will have the following form according to (2.7)

$$\sum_{n=1}^{2}(-1)^{n-1} b_n \,\mathrm{Im}\,\sigma_n \sqrt{\lambda_n(1-\lambda_n)} = 0$$

Let us note that the problem of the deformation of a half-plane with an oblique crack is solved in /12/ for one particular kind of load.

3. It is known that the stress field singularities can occur also at smooth points of the boundary even if the boundary data are limited, for instance, when p_θ and $u_{\theta*}'$ at a smooth point of the boundary have a finite discontinuity. In this case $w_n'(s)$ has a finite discontinuity at the point s_0 and since $\Psi_n'(\zeta_n) = w_n'(s)h_n(\zeta_n)/z_n'(s)$, then the integral in (1.14) will have a logarithmic singularity /8/

$$\Phi_n' = (-1)^{n-1}\frac{g_{n0}}{2\pi i t_n(\theta)} \times \left(\frac{w_n'(s_0-0)-w_n'(s_0+0)}{z_n'(s_0)}\ln(\zeta_n(z_n)-\zeta_{n0})+\Psi_{n0}\right)$$

where Ψ_{n0} has a finite limit as $s \to s_0$. The stresses in (1.1) will also have the same singularity.

If the finite discontinuity has an integral force, corresponding to a concentrated load, then Φ_n has a logarithmic singularity

$$\Phi_n = (-1)^{n-1}\frac{w_n(s_0-0)-w_n(s_0+0)}{2\pi i t_n(\theta)}\ln(\zeta_n(z_n)-\zeta_{n0})+\Phi_{n0}$$

while the derivative Φ_n', meaning the stresses, have a singularity of order one. Let us note that this singularity is conserved even at the corner point. Indeed, in this case $(\zeta_n-\zeta_{n0})^{\nu_n} = (z_n-z_{n0})/g_n(\zeta_n)$ and $\nu_n\ln(\zeta_n-\zeta_{n0}) = \ln(z_n-z_{n0})-\ln g_n(\zeta_n)$, therefore

$$\Phi_n' = \frac{(-1)^{n-1}}{2\pi i t_n(\theta)}\,\frac{w_n(s_0-0)-w_n(s_0+0)}{\nu_n(z_n-z_{n0})}+\Psi_{n0}$$

The singularities can occur for the stresses at smooth points even when a "discontinuity" of the nature of the boundary conditions occurs at these points.

Let us consider the following boundary value problem for the case of the roots $\gamma_n = i\beta_n$. On the section (a, b) of the boundary of the domain D let be given p_1 and u_2, but p_2 and u_1 are given outside $[a, b]$, while p_k and u_k' are bounded and Hölder-continuous outside the ends of the section. We have the following boundary value problem /5/ for the functions Φ_n :

$$2\operatorname{Re}\Phi_n = (-1)^n u_{2n}(A_n^{-1}z_n) \quad \text{on} \quad (a_n, b_n)$$
$$2\operatorname{Im}\Phi_n = (-1)^n u_{1n}(A_n^{-1}z_n) \;\text{out of}\; [a_n, b_n]$$
$$u_{kn} = (\varkappa_{3-n}P_k - u_{3-k})/(\varkappa_2-\varkappa_1)$$

where (a_n, b_n) is the image of (a, b) for the mapping A_n. Mapping D_n on a half-plane, we arrive at an analogous problem for $\Phi_n(z_n(\zeta_n))$ which is solved by using the Keldysh–Sedov formula. In this case the solution has the form /8/:

$$\Phi_n = \frac{1}{g_n(\zeta_n)}\left(\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{g_n(\tau_n)v_n(A_n^{-1}z_n(\tau_n))}{\tau_n-\zeta_n}d\tau_n + c_{n0}+c_{n1}\zeta_n+c_{n2}\zeta_n^2\right)$$
$$v_n = (-1)^n u_{2n} \quad \text{on} \quad (a_n, b_n),\; \beta_n v_n = (-1)^n i u_{1n} \quad \text{out of} \quad [a_n, b_n]$$
$$g_n(\zeta_n) = ((\zeta_n-\zeta_n^a)(\zeta_n-\zeta_n^b))^{1/2}$$

where c_{nj} $(j = 0, 1, 2)$ are arbitrary constants, and ζ_n^a, ζ_n^b are prototypes of the points a_n, b_n.

This solution can have a singularity of order 1/2 as Φ_n' and then $z \to z_0$ and the stress field have a singularity of order 3/2.

A solution bounded at all ends is possible for certain constraints on the data of the problem. However, even in this case the singularity of the stress field can be of order 1/2 for $z \to z_0$.

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Translated by M.D.F.